

An effective theory approach to unstable particles

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(Dated: December 19, 2003)

Using the hierarchy of scales between the mass, M and the width Γ of a heavy, unstable particle we construct an effective theory that allows calculations for resonant processes to be systematically expanded in powers of the coupling α and Γ/M . We illustrate the method by computing the next-to-leading order line shape of a scalar resonance in an abelian gauge-Yukawa model.

PACS numbers: 11.80.Cr

Higher-order calculations for processes involving massive, unstable particles close to resonance suffer from the breakdown of ordinary perturbation theory, since the intermediate propagator becomes singular. This singularity is avoided if the finite width, Γ , of the unstable particle is taken into account in the construction of the propagator via resummation of self-energy insertions. There are a number of approaches along this line to avoid the problem [1]. However, so far there is no method that allows to systematically improve the accuracy of calculations order by order in perturbation theory.

The purpose of this letter is to present such a method. We are concerned with processes involving an unstable particle close to resonance. The main idea is to exploit the hierarchy of scales $\Gamma \ll M$, where M is the pole mass, in order to systematically organise the calculations in a series in the coupling, α , and Γ/M . While the expansion in α is obvious, we construct an effective theory to perform the expansion in Γ/M . The main idea of our approach is very similar to non-relativistic QCD, where an expansion in α and the velocity of the heavy quarks is made and a first step in this direction has been presented in [2]. We will identify all relevant modes and use them to write the operators of the Lagrangian of the effective theory. This Lagrangian is then matched to the underlying theory, using the method of regions [3]. In this letter we will outline the basic idea and we refer to [4] for more details.

Let us illustrate the method with a toy model that involves a massive scalar field, ϕ , and two fermion fields. The scalar as well as one of the fermion fields, ψ , (the “electron”) are charged under an abelian gauge symmetry, whereas the other fermion, χ , (the “neutrino”) is neutral. The model allows for the scalar to decay into an electron-neutrino pair through a Yukawa interaction. We also include a scalar self-interaction to ensure renormalisability. The Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & (D_\mu \phi)^\dagger D^\mu \phi - \hat{M}^2 \phi^\dagger \phi + \bar{\psi} i \not{D} \psi + \bar{\chi} i \not{\partial} \chi \\ & - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \\ & + y \phi \bar{\psi} \chi + y^* \phi^\dagger \bar{\chi} \psi - \frac{\lambda}{4} (\phi^\dagger \phi)^2 + \mathcal{L}_{\text{ct}}, \end{aligned} \quad (1)$$

The fields and parameters are renormalised in a particular scheme, to be specified later. \hat{M} and \mathcal{L}_{ct} denote the corresponding renormalised mass and counterterm Lagrangian. We define $\alpha_g \equiv g^2/(4\pi)$, $\alpha_y \equiv (yy^*)/(4\pi)$ and assume $\alpha_g \sim \alpha_y \sim \alpha$. For the counting in the following we will assume $\alpha_\lambda \equiv \lambda/(4\pi) \sim \alpha^2/(4\pi)$.

We would like to obtain the totally inclusive cross section for the process

$$\bar{\nu}(q) + e^-(p) \rightarrow X \quad (2)$$

as a function of $s \equiv (p+q)^2$ by calculating the forward scattering amplitude $\mathcal{T}(s)$ and taking its imaginary part. In particular, we are interested in the region $s \approx M^2$, or more precisely $s - M^2 \sim M\Gamma \sim \alpha M^2 \ll M^2$. In this kinematic region the cross section is enhanced due to the propagator of the scalar. Furthermore, at each order in α we get additional contributions proportional to $\alpha \hat{M}^2/(s - \hat{M}^2) \sim 1$ due to self-energy insertions.

As stated above, our approach is based on the hierarchy of scales $\Gamma \ll M$. Thus, we systematically expand the cross section in powers of α and

$$\delta \equiv \frac{s - \hat{M}^2}{\hat{M}^2} \sim \frac{\Gamma}{M} \quad (3)$$

In a theory that formulates this expansion correctly, other issues like resummation of self-energy insertions and gauge invariance are taken care of automatically.

Before turning to the formulation of such a theory, we remark that the total cross section (2) has an initial state collinear singularity which has to be absorbed into the electron distribution function. In what follows it is understood that this singularity is subtracted minimally.

We now turn to the main part of this letter and discuss how to construct the effective theory. In a first step we integrate out hard momenta $k \sim M$. The effective theory will then not contain any longer dynamical hard modes since their effect is included in the coefficients of the operators. The hard effects are associated with the factorisable corrections, whereas the effects of the still dynamical modes corresponds to the non-factorisable corrections [2]. On the level of Feynman diagrams, this amounts to using the method of regions to separate loop integrals into

various contributions [3]. The hard part is obtained by expanding the integrand in δ . The difference between the full integral and its hard part has to be reproduced by modes corresponding to momentum configurations that are near mass-shell. The main task is to identify these modes, write the operators of the effective Lagrangian in terms of the corresponding field operators and then compute the coefficients of the operators by matching (up to a certain order in α and δ).

Our goal is to carry out this programme for our model to an order in α and δ that is sufficient to compute $\mathcal{T}^{(0)} + \mathcal{T}^{(1)}$, the forward scattering amplitude at next-to-leading order (NLO). The basic process under consideration is the following: we start with highly energetic fermions, produce a near mass-shell scalar which then decays again into highly energetic fermions. Accordingly we split the effective Lagrangian into three parts. The first, $\mathcal{L}_{\text{HSET}}$, describes the heavy scalar field near mass-shell and its interaction with the gauge field. The second part, $\mathcal{L}_{\text{SCET}}$, describes energetic (charged) fermions and their interactions with the gauge field. Finally, the third part, \mathcal{L}_{int} , describes the external fermions and how they interact to produce the final state. We will discuss these three parts in turn.

The construction of the effective Lagrangian, $\mathcal{L}_{\text{HSET}}$, follows closely the construction of the effective Lagrangian for heavy quark effective theory (HQET) [5]. We write the momentum of the scalar particle near resonance as $P = \hat{M}v + k$, where the velocity vector v satisfies $v^2 = 1$ and the residual momentum k scales as $M\delta$. We will call such a scalar field a “soft” field (in [2] the term “resonant” has been used). Thus, for a soft scalar field we have $P^2 - \hat{M}^2 \sim M\delta$ and this remains true if the scalar particle interacts with a soft gauge boson with momentum $\sim M\delta$. In analogy to HQET we remove the rapid spatial variation $e^{-i\hat{M}v \cdot x}$ from the ϕ -field and define

$$\phi_v(x) \equiv e^{i\hat{M}v \cdot x} \mathcal{P}_+ \phi(x). \quad (4)$$

where \mathcal{P}_+ projects onto the positive frequency part to ensure that ϕ_v is a pure destruction field. We now write the effective Lagrangian in terms of ϕ_v and construct the bilinear terms so as to reproduce the two-point function close to resonance. Denoting the complex pole of the propagator by \bar{s} and the residue at the pole by R_ϕ the propagator can be written as

$$\frac{i R_\phi}{P^2 - \bar{s}} = \frac{i R_\phi}{2\hat{M}vk + k^2 - (\bar{s} - \hat{M}^2)}. \quad (5)$$

We define the matching coefficient

$$\Delta \equiv \frac{\bar{s} - \hat{M}^2}{\hat{M}} \quad (6)$$

and $a_\perp^\mu \equiv a^\mu - (va)v^\mu$ for any vector and express the solution $P^2 = \bar{s}$ expanded in δ as

$$(vk) = -\hat{M} + \sqrt{\hat{M}^2 + \hat{M}\Delta - k_\perp^2}$$

$$= \frac{\Delta}{2} - \frac{\Delta^2 + 4k_\perp^2}{8\hat{M}} + \mathcal{O}(\delta^3) \quad (7)$$

Therefore, the bilinear terms are given by

$$\begin{aligned} \mathcal{L}_{\phi\phi} = & 2\hat{M}\phi_v^\dagger \left(iv \cdot D_s - \frac{\Delta}{2} \right) \phi_v \\ & + 2\hat{M}\phi_v^\dagger \left(\frac{(iD_{s\perp})^2}{2\hat{M}} + \frac{\Delta^2}{8\hat{M}} \right) \phi_v + \dots, \end{aligned} \quad (8)$$

where $D_s \equiv \partial - igA_s$ denotes the covariant derivative with a soft photon field. In obtaining $\mathcal{L}_{\phi\phi}$ we exploited the fact that the gauge invariance of the full Lagrangian is not broken by the separation into hard and soft parts. Therefore, the effective Lagrangian must be gauge invariant as well and we can obtain the interaction of the scalar with the soft photon simply by replacing $\partial \rightarrow D_s$. The gauge invariance of Δ follows from the gauge invariance of \bar{s} and \hat{M} . Furthermore, Δ is given entirely by hard contributions, which justifies its interpretation as matching coefficient. Using (6) we can express it in terms of the hard part of the self-energy $\Pi_h(s)$. Writing $\Pi_h(s) = \hat{M}^2 \sum_{k,l} \delta^l \Pi^{(k,l)}$, where it is understood that $\Pi^{(k,l)} \sim \alpha^k$, we obtain

$$\Delta \equiv \sum_i \Delta^{(i)} = \hat{M} \Pi^{(1,0)} + \hat{M} \left(\Pi^{(2,0)} + \Pi^{(1,1)} \Pi^{(1,0)} \right) + \dots \quad (9)$$

Explicit results for $\Delta^{(1)}$ and $\Delta^{(2)}$ in the $\overline{\text{MS}}$ and pole renormalisation scheme can be found in [4]. Here we only note that in the pole scheme $\Delta^{(1)} = -i\Gamma$. Inserting the expansion (9) into (8) and supplementing $\mathcal{L}_{\phi\phi}$ with the kinetic terms for soft photons and fermions we obtain

$$\begin{aligned} \mathcal{L}_{\text{HSET}} = & 2\hat{M}\phi_v^\dagger \left(iv \cdot D_s - \frac{\Delta^{(1)}}{2} \right) \phi_v \\ & + 2\hat{M}\phi_v^\dagger \left(\frac{(iD_{s\perp})^2}{2\hat{M}} + \frac{[\Delta^{(1)}]^2}{8\hat{M}} - \frac{\Delta^{(2)}}{2} \right) \phi_v \\ & - \frac{1}{4} F_{s\mu\nu} F_s^{\mu\nu} + \bar{\psi}_s i \not{D}_s \psi_s + \bar{\chi}_s i \not{\partial} \chi_s \end{aligned} \quad (10)$$

Each term in $\mathcal{L}_{\text{HSET}}$ can be assigned a scaling power in δ . In momentum space the propagator of the ϕ_v field scales as $1/\delta$. Because $\int d^4k$ counts as δ^4 the soft scalar field $\phi_v(x)$ scales as $\delta^{3/2}$. Since $\Delta^{(1)} \sim D_s \sim M\delta$ both terms in the first line of (8) scale as δ^4 and are leading terms. The terms in the second line are suppressed by one power in δ or α . Finally, since A_s^μ scales as δ and the soft fermion fields scale as $\delta^{3/2}$ the terms in the last line of (8) scale as δ^4 . In (8) we have left out terms further suppressed in δ or α . As we will see, they are not needed for the calculation of the line shape at NLO. However, we stress that the expansion can be performed to whatever accuracy is needed.

As a final remark related to the construction of $\mathcal{L}_{\text{HSET}}$ we note that computing the scalar propagator to all orders in δ using $\mathcal{L}_{\text{HSET}}$ does not reproduce (5). Instead

near resonance we obtain $i\varpi^{-1}R_{\text{eff}\phi}/(P^2 - \bar{s})$, where $\varpi^{-1} \equiv \sqrt{\hat{M}^2 + \hat{M}\Delta - k_+^2}/\hat{M} = 1 + \mathcal{O}(\delta, \alpha)$. The difference in the residue factor will be taken care of automatically by loop graphs in the effective theory. The difference in the normalization due to the factor ϖ^{-1} however has to be taken into account by an additional wave-function normalisation factor $\varpi^{-1/2}$ for each external ϕ_v -line in the effective theory.

Next, we turn to the construction of the effective Lagrangian, $\mathcal{L}_{\text{SCET}}$, associated with energetic fermions. We need a “collinear” mode to describe a fermion with large momentum in the say \vec{n}_- direction. Such modes have been discussed previously within the context of soft-collinear effective theory (SCET) [6]. In particular, in position space the Lagrangian has been worked out to order δ^2 in [7] and we can simply take the parts relevant to us from there. We mention that we call “soft” here what is usually called “ultrasoft” in the context of SCET and in the power counting our δ corresponds to λ^2 in [7]. For each direction defined by an energetic particle we introduce two reference light-like vectors, n_{\pm} , with $n_+^2 = n_-^2 = 0$ and $n_+ n_- = 2$ and we write the corresponding momentum as

$$p^\mu = (n_+ p) \frac{n_+^\mu}{2} + p_\perp^\mu + (n_- p) \frac{n_-^\mu}{2}, \quad (11)$$

where $n_+ p \sim M$, $n_- p \sim M\delta$ and $p_\perp \sim M\delta^{1/2}$. Given a certain direction n_- we introduce the collinear field ψ_c which satisfies $\not{n}_- \psi_c = 0$. The terms relevant for the calculation of $\mathcal{T}^{(0)} + \mathcal{T}^{(1)}$ are then given by

$$\mathcal{L}_{\text{SCET}} = \bar{\psi}_c \left(in_- D + i \not{D}_\perp \frac{1}{in_+ D_c + i\epsilon} i \not{D}_\perp c \right) \frac{\not{n}_+}{2} \psi_c \quad (12)$$

Since we are concerned with the forward scattering amplitude, the only directions defined by energetic particles are given by the incoming electron and (anti)neutrino. Thus, we have two sets of collinear modes, one for the incoming electron, ψ_{c1} , and one for the incoming (anti)neutrino, χ_{c2} . Of course, in the case of the neutrino, the covariant derivatives in (12) have to be replaced by ordinary derivatives.

Following [7] we note that all terms in (12) scale as δ^4 . Terms of order $\delta^{9/2}$ and δ^5 do exist, but they are not needed for our application, since they would result in contributions suppressed by an additional power of α and, therefore, contribute only at NNLO. But we stress again that there is no difficulty in going to higher orders in the expansion if needed.

The last part to consider is \mathcal{L}_{int} . It has to include operators that allow the production and decay of the unstable particle. Without introducing additional modes it is not possible to include such vertices as ordinary interaction terms in the effective Lagrangian [4]. The reason is that the momenta of generic collinear states ψ_{c1} and $\bar{\chi}_{c2}$ do

not add up to a momentum of the form $P = Mv + k$. Either we have to implement this kinematic constraint on our external states by hand [4] or we have to introduce a new “external collinear” mode. Taking the second option, we define an external collinear mode with large momentum in the \vec{n}_- direction by assigning it a momentum $\hat{M}n_-/2 + k$, where $k \sim \delta$. This mode has the same virtuality $\hat{M}\delta^{1/2}$ as an ordinary collinear mode but the momentum is not given by (11). It has a fixed large component such that the two incoming fermions produce a scalar near mass shell. All other components scale as δ . For such a mode it is useful to extract the fixed large momentum and define

$$\psi_{n_-}(x) \equiv e^{i\hat{M}/2(n_-x)} \psi_{c1}(x) \quad (13)$$

and similarly for χ_{n_+} . For the purpose of computing $\mathcal{T}^{(0)} + \mathcal{T}^{(1)}$ it is sufficient to take the first term of $\mathcal{L}_{\text{SCET}}$, (12), with a soft photon only to describe the interaction of the external collinear fermions with the photons

$$\mathcal{L}_{\pm} = \bar{\psi}_{n_-}(in_- D_s) \frac{\not{n}_+}{2} \psi_{n_-} + \bar{\chi}_{n_+}(in_+ \partial) \frac{\not{n}_-}{2} \chi_{n_+} \quad (14)$$

Using the external modes we can implement the production and decay vertices as interaction terms in \mathcal{L}_{int} . Because adding soft fields results in a further suppression in δ we can restrict ourselves to

$$\begin{aligned} \mathcal{L}_{\text{int}} = & C y \phi_v \bar{\psi}_{n_-} \chi_{n_+} + C y^* \phi_v^\dagger \bar{\chi}_{n_+} \psi_{n_-} \\ & + F y y^* (\bar{\psi}_{n_-} \chi_{n_+}) (\bar{\chi}_{n_+} \psi_{n_-}) \end{aligned} \quad (15)$$

where $C = 1 + \mathcal{O}(\alpha)$ and F are the matching coefficients. The external fields scale as $\delta^{3/2}$. Thus, an insertion of a $\phi\psi\chi$ operator results in $\int d^4x \phi_v \bar{\psi}_{n_-} \chi_{n_+} \sim \delta^{1/2}$. The forward scattering amplitude can be obtained by two insertions of this operator. Taking into account the scaling of the external state $\langle \bar{\nu} e^- | \sim \delta^{-1}$ we see that $\mathcal{T}^{(0)} \sim \alpha/\delta$. The four-fermion operator is suppressed in δ and results in a contribution $\sim \alpha$ to \mathcal{T} . Thus, to compute $\mathcal{T}^{(1)}$ we need $C^{(1)}$, the $\mathcal{O}(\alpha)$ contribution to the matching coefficient C , while F is only needed at tree level.

The coefficient $C^{(1)}$ is obtained by matching the on-shell three-point function of a scalar field, an electron and a neutrino at order $y\alpha$ and at leading order in δ . In particular, this involves the computation of (the hard part) of the vertex diagram and the additional wave-function normalisation factor $\varpi^{-1/2}$ mentioned above has to be taken into account. For the precise matching equation as well as the explicit result for $C^{(1)}$ we refer to [4]. Here it suffices to say that these are completely standard loop calculations. To obtain $F^{(0)}$ we have to match the four-point function at tree level, but include subleading terms in δ . The explicit result is $F^{(0)} = 1/4\hat{M}^2$.

We have now completed the construction of the effective Lagrangian $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{HSET}} + \mathcal{L}_{\pm} + \mathcal{L}_{\text{int}}$ to a sufficient accuracy to compute \mathcal{T} at NLO. At leading order there is

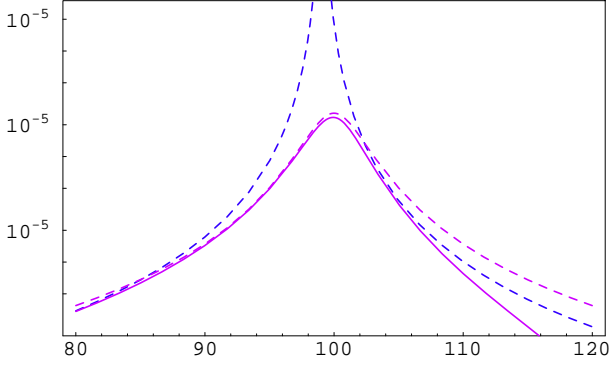


FIG. 1: The line shape (in GeV^{-2}) in the effective theory at LO (light grey/magenta dashed) and NLO (light grey/magenta) and the LO cross section off resonance in the full theory (dark grey/blue dashed) as a function of the centre-of-mass energy.

only one diagram, involving two three-point vertices and one resonant scalar propagator. We get

$$i\mathcal{T}^{(0)} = \frac{i^3 yy^*}{2\hat{M}\mathcal{D}} [\bar{u}(p)v(q)] [\bar{v}(q)u(p)]. \quad (16)$$

where we defined $\mathcal{D} \equiv \sqrt{s} - \hat{M} - \Delta^{(1)}/2$. Within the effective theory there are three classes of diagrams that contribute to $\mathcal{T}^{(1)}$. Firstly, there are hard corrections consisting of a propagator insertion $[\Delta^{(1)}]^2/4 - \hat{M}\Delta^{(2)}$ as well as a vertex insertion $C^{(1)}$. Secondly, there is a four-point vertex diagram due to the $(\bar{\psi}\chi)(\bar{\chi}\psi)$ operator in \mathcal{L}_{int} . The third class are soft-photon loop diagrams, corresponding to the non-factorizable corrections. There are four such diagrams, a correction to the scalar propagator, two vertex correction diagrams and a box diagram. Adding up all these contributions and using the explicit result for $C^{(1)}$ (in the $\overline{\text{MS}}$ -scheme) [4] we obtain

$$i\mathcal{T}^{(1)} = i\mathcal{T}^{(0)} \times \quad (17)$$

$$\left[a_g \left(3 \ln \frac{-2\hat{M}\mathcal{D}}{\nu^2} + 4 \ln \frac{-2\hat{M}\mathcal{D}}{\hat{M}^2} \ln \frac{-2\hat{M}\mathcal{D}}{\nu^2} - 7 \ln \frac{-2\hat{M}\mathcal{D}}{\hat{M}^2} - \frac{3}{2} \ln \frac{\hat{M}^2}{\mu^2} - \frac{7}{2} + \frac{2\pi^2}{3} \right) + a_y \left(2 \ln \frac{\hat{M}^2}{\mu^2} - \frac{1}{2} - i\pi \right) - \frac{[\Delta^{(1)}]^2}{8\hat{M}\mathcal{D}} + \frac{\Delta^{(2)}}{2\mathcal{D}} - \frac{\mathcal{D}}{2\hat{M}} \right]$$

where we have subtracted the initial state collinear singularities minimally. We denote the corresponding factorisation scale by ν to distinguish it from the renormalisation scale μ .

We can now perform the polarisation average and take the imaginary part of $(\mathcal{T}^{(0)} + \mathcal{T}^{(1)})/s$. This result will describe the line shape near resonance with a relative accuracy of α^2 . Moving away from the resonance, the relative error becomes of order unity, since δ is not small

any longer. To obtain a good description for all values of \sqrt{s} , the result of the effective theory has to be matched to the off-resonance result of the full theory.

In Fig 1 we show the leading order line shape in the effective theory and the tree-level (order α^2) cross section off resonance in the full theory. The two results do agree in an intermediate region where both calculations are valid. This allows to obtain a consistent LO result for all values of \sqrt{s} . We also show the NLO line shape. For the numerical results we have chosen to use the $\overline{\text{MS}}$ -scheme with $\alpha_y = \alpha_g = 0.1$ and $\alpha_\lambda = (0.1)^2/(4\pi)$. The pole mass is assumed to be $M = 100$ GeV which results in the $\overline{\text{MS}}$ value $\hat{M} = 98.8$ GeV for the LO result and $\hat{M} = 99.1$ GeV for the NLO result. Furthermore, we have chosen a variable factorisation scale such that there are no large logarithms involving ν . We remark that in order to obtain an improved NLO result for the whole region of \sqrt{s} , the NLO line shape would have to be matched to the NLO off-resonance cross section in the full theory.

The example considered here is based on a rather simple toy model. Nevertheless, it allows to address the conceptual issues related to unstable particles. The main result is that, using an effective theory approach, calculations can be performed in a systematic way in expanding in the small quantities α and Γ/M . Applying our method to the Standard Model might require more tedious calculations, but the main result remains valid.

The work of M.B. and A.P.C is supported in part by the DFG Sonderforschungsbereich/Transregio 9 “Computergestützte Theoretische Teilchenphysik”

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